Nonhomogeneous equation

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q, g are continuous functions on an open interval I.

- The associated homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0$$

Variation of Parameters

- In this section we will learn the **variation of parameters method** to solve the nonhomogeneous equation. As with the method of undetermined coefficients, this procedure relies on knowing solutions to homogeneous equation.

- Variation of parameters is a general method, and requires no detailed assumptions about solution form. However, certain integrals need to be evaluated, and this can present difficulties.

Example

Find a particular solution of

$$y'' + 9y = 2\csc t$$

- We cannot use method of undetermined coefficients since $g(t)$ is a quotient of $\sin t$ or $\cos t$, instead of a sum or product.
- Recall that the solution to the homogeneous equation is

$$y_C(t) = c_1 \cos 3t + c_2 \sin 3t$$

- To find a particular solution to the nonhomogeneous equation, we begin with the form

$$y(t) = u_1(t)\cos 3t + u_2(t)\sin 3t$$

- Then

$$y'(t) = u_1'(t)\cos 3t - 3u_1(t)\sin 3t + u_2'(t)\sin 3t + 3u_2(t)\cos 3t$$

or $$y'(t) = -3u_1(t)\sin 3t + 3u_2(t)\cos 3t + u_1'(t)\cos 3t + u_2'(t)\sin 3t$$

(1) Derivatives

- From the previous slide,

$$y'(t) = -3u_1(t)\sin 3t + 3u_2(t)\cos 3t + u_1'(t)\cos 3t + u_2'(t)\sin 3t$$

- Note that we need two equations to solve for u_1 and u_2. The first equation is the differential equation. To get a second equation, we will require

$$u_1'(t)\cos 3t + u_2'(t)\sin 3t = 0$$

- Then

$$y'(t) = -3u_1(t)\sin 3t + 3u_2(t)\cos 3t$$

- Next,

$$y''(t) = -3u_1'(t)\sin 3t - 9u_1(t)\cos 3t + 3u_2'(t)\cos 3t - 9u_2(t)\sin 3t$$

(2) Two equations

- Recall that our differential equation is

$$y'' + 9y = 2\csc t$$

- Substituting y'' and y into this equation, we obtain

$$-3u_1'(t)\sin 3t - 9u_1(t)\cos 3t + 3u_2'(t)\cos 3t - 9u_2(t)\sin 3t$$
$$+9\big(u_1(t)\cos 3t + u_2(t)\sin 3t\big) = 2\csc t$$

- This equation simplifies to

$$-3u_1'(t)\sin 3t + 3u_2'(t)\cos 3t = 2\csc t$$

- Thus, to solve for u_1 and u_2, we have the two equations:

$$-3u_1'(t)\sin 3t + 3u_2'(t)\cos 3t = 2\csc t$$
$$u_1'(t)\cos 3t + u_2'(t)\sin 3t = 0$$

(3) Solve u_1

- To find u_1 and u_2 , we need to solve the equations

$$-3u_1'(t)\sin 3t + 3u_2'(t)\cos 3t = 2\csc t$$

$$u_1'(t)\cos 3t + u_2'(t)\sin 3t = 0$$

- From second equation,

$$u_2'(t) = -u_1'(t)\frac{\cos 3t}{\sin 3t}$$

- Substituting this into the first equation,

$$-3u_1'(t)\sin 3t + 3\left[-u_1'(t)\frac{\cos 3t}{\sin 3t}\right]\cos 3t = 3\csc t$$

$$-3u_1'(t)\sin^2(3t) - 3u_1'(t)\cos^2(3t) = 2\csc t\sin 3t$$

$$-3u_1'(t)\left[\sin^2(3t) + \cos^2(3t)\right] = 2\left[\frac{3\sin t - 4\sin^3 t}{\sin t}\right]$$

$$u_1'(t) = -\frac{2}{3}\left[3 - 4\sin^2 t\right]$$

(4) Solve u_2

- From the previous slide,

$$u_1'(t) = -\frac{2}{3}[3-4\sin^2 t], \quad u_2'(t) = -u_1'(t)\frac{\cos 3t}{\sin 3t}$$

- Then

$$u_2'(t) = -\frac{2}{3}[3-4\sin^2 t]\cdot\left[\frac{\cos 3t}{\sin 3t}\right] = -\frac{2}{3}[3-4\sin^2 t]\left[\frac{4\cos^3 t - 3\cos t}{3\sin t - 4\sin^3 t}\right]$$

$$= -\frac{2}{3}\left[\frac{4\cos^3 t}{\sin t} - \frac{3\cos t}{\sin t}\right] = -\frac{8}{3}\cos t\cdot\cos^2 t + 2\cot t$$

- Thus

$$u_1(t) = \int u_1'(t)dt = \int -\frac{2}{3}[3-4\sin^2 t]dt = -\frac{2}{3}t + \frac{2}{3}\sin 2t + c_1$$

$$u_2(t) = \int u_2'(t)dt = \int\left(-\frac{8}{3}\cot t\cdot\cos^2 t + 2\cot t\right)dt = -\frac{2}{3}\cos 2t - \frac{8}{3}\ln|\sin t| + c_2$$

(5) General solution

- Recall our equation and homogeneous solution y_C :

$$y'' + 9y = 2\csc t, \quad y_C(t) = c_1 \cos 3t + c_2 \sin 3t$$

- Using the expressions for u_1 and u_2 on the previous slide, the general solution to the differential equation is

$$\begin{aligned} y(t) &= u_1(t)\cos 3t + u_2(t)\sin 3t + y_C(t) \\ &= \left(-\frac{2}{3}t + \frac{2}{3}\sin 2t\right)\cos 3t + \left(-\frac{2}{3}\cos 2t - \frac{8}{3}\ln|\sin t|\right)\sin 3t + y_C(t) \\ &= -\frac{2}{3}t\cos 3t + \frac{2}{3}\sin 2t\cos 3t - \frac{2}{3}\cos 2t\sin 3t - \frac{8}{3}\ln|\sin t|\sin 3t \\ &\quad + c_1\cos 3t + c_2\sin 3t. \end{aligned}$$

Theorem

- Consider the equations

$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

- If the functions p, q, and g are continuous on an open interval I, and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt$$

and the general solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Summary

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Suppose y_1, y_2 are fundamental solutions to the homogeneous equation associated with the nonhomogeneous equation above, where we note that the coefficient on y'' is 1.
- To find u_1 and u_2, we need to solve the equations

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

- Doing so, and using the Wronskian, we obtain

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

- Thus

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt + c_1, \quad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt + c_2$$